JOURNAL OF APPROXIMATION THEORY **28**, 83–86 (1980)

A Remark on the Lower Semi-Continuity of the Set-Valued Metric Projection

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Communicated by E. W. Cheney

Received July 31, 1978

During the last 10 years, a series of papers have been concerned with continuity properties of the set-valued metric projection from a normed linear space to a proximal linear subspace. The metric projection is the mapping defined by

$$E \to 2^G,$$
$$x \mapsto P_G(x) := \{g \in G: \| x - g \| = d(x, G)\},$$

where E is a normed linear space and $G \subset E$ a proximial linear subspace. In this context, lower and upper semi-continuity and continuous selections are three of the most important notions: Let F: $E \to 2^G$ be a set-valued mapping (E, G Hausdorff-spaces.)

F is called lower semi-continuous (l.s.c.), iff, for each $x \in E$ and each open subset $U \subset G$ with $U \cap F(x) \neq \varnothing$, there exists a neighborhood $V \subset E$ of x such that $U \cap F(y) \neq \varnothing$ for each $y \in V$.

F is called upper semi-continuous (u.s.c.), iff, for each $x \in E$ and each open subset $U \subset G$ with $F(x) \subset U$, there exists a neighborhood $V \subset E$ of x with $F(y) \subset U$ for each $y \in V$.

A single-valued mapping s: $E \to G$ is called a selection for F, iff $s(x) \in F(x)$ for each $x \in E$.

A well-known theorem of Michael [4, Theorem 3.2″] states that an l.s.c. set-valued mapping from a paracompact Hausdorff-space into a Banach-space with closed convex images always admits a continuous selection.

In his paper [5], Nürnberger introduces the following notion:

Let E be a normed linear space and $G \subset E$ a proximinal linear subspace. A selection s: $E \to G$ for P_G is said to have the "Nulleigenschaft," iff $s(x) = 0$ for each $x \in E$ with $0 \in P_G(x)$.

Using this notion, the lower semi-continuity of the set-valued metric projection can be characterized as follows:

0021-9045/80/010083–04$02.00/0

1. PROPOSITION. *Let E be a normed linear space, $G \subset E$ a proximal linear subspace; compare the following two statements*:

(a) *P_G admits a continuous selection with the "Nulleigenschaft"*

(b) *P_G is* l.s.c.

We have (a) $\Rightarrow$ (b) *and if G is complete, also* (b) $\Rightarrow$ (a).

Proof. (a) $\Rightarrow$ (b) Let $x \in E$ and $U \subset G$ be an open subset with $P_G(x) \cap U \neq \varnothing$, furthermore let $g \in P_G(x) \cap U$; we have $0 \in P_G(x - g) \cap (U - g)$ and hence $s(x - g) = 0$. Now, let $V \subset E$ be a neighborhood of x with $s(y - g) \in U - g$ for each $y \in V$. It follows $s(y - g) + g \in P_G(y) \cap U$ and therefore $P_G(y) \cap U \neq \varnothing$ for each $y \in V$.

(b) $\Rightarrow$ (a) Note that $P_G^{-1}(0) := \{x \in E: 0 \in P_G(x)\}$ is a closed subset of E; hence, the set-valued mapping

$$\begin{aligned} &E \to 2^G \\ &x \mapsto \{0\} \qquad \text{if } x \in P_G^{-1}(0) \\ &\quad \mapsto P_G(x) \qquad \text{if } x \notin P_G^{-1}(0) \end{aligned}$$

is l.s.c. . The rest is a consequence of the abovementioned theorem of Michael [4].

The following two older results relate this criterion with the situation in concrete spaces:

2. THEOREM. (Blatter *et al.* [1, Theorem 2]). *Let Q be a compact Hausdorff-space; $G \subset C(Q)$ a proximal linear subspace such that $P_G(x)$ is a finite-dimensional subset of G for each $x \in C(Q)$. Compare the following two statements*:

(a) *P_G is* l.s.c.

(b) *$ZP_G(x) := \{q \in Q: g(q) = 0$ f.e. $g \in P_G(x)\}$ is an open set for each $x \in P_G^{-1}(0)$.*

We have (a) $\Rightarrow$ (b) *and if P_G is* u.s.c., *also* (b) $\Rightarrow$ (a).

This theorem has been generalized by Brosowski and Wegmann [2] to spaces of the type $C_0(Q, X)$ where Q is a locally compact Hausdorff-space and X is a strictly normed linear space ($C_0(Q, X)$ is the space of all continuous functions from Q to X vanishing at infinity).

3. THEOREM (Lazar *et al.* [3, Theorem 1.1]). *Let (T, μ) be a σ-finite measure space, $G \subset L_1(T, \mu)$ an n-dimensional linear subspace ($n \in \mathbb{N}$). The following two statements are equivalent*:

(a) *P_G is* l.s.c.

(b) *There does not exist an $f \in G^{\perp}\backslash\{0\} \subset L_{\infty}(T, \mu)$ and a $g \in G$ with*

(o) $S(f) := \{t \in T: |f(t)| < \|f\|\}$ *is the union of at most* $(n - 1)$ *atoms*

(i) $S(f) \subset Z(g) := \{t \in T: g(t) = 0\}$

(ii) supp(g) *is not the union of a finite number of atoms.*

(*All relations between subsets of T are to be understood modulo a set with measure zero*).

A part of the paper of Nürnberger [5] is concerned with developing criteria for the existens of continuous selections with the "Nulleigenschaft." We are now able to give this alternative approach to his results:

4. COROLLARY. *Let Q be a compact Hausdorff-space, $G \subset C(Q)$ a proximal linear subspace such that for each $x \in C(Q)$, $P_G(x)$ is a finite-dimensional set. Compare the following two statements*:

(a) *P_G admits a continuous selection with the "Nulleigenschaft"*

(b) *$ZP_G(x)$ is an open set for each $x \in P_G^{-1}(0)$.*

We have (a) $\Rightarrow$ (b) *and if P_G is* u.s.c., *also* (b) $\Rightarrow$ (a).

Proof. This is another formulation of Theorem 3.

5. COROLLARY. *Let Q be a compact Hausdorff-space, $G \subset C(Q)$ a proximal linear subspace such that $P_G(x)$ is a finite-dimensional set for each $x \in C(Q)$; moreover, let P_G be* u.s.c.; *then*:

(1) *P_G admits a continuous selection if it is* l.s.c.

(2) *P_G is* l.s.c., *if it admits a continuous selection with the "Nulleigenschaft."*

Proof. This is a very weak form of Proposition 1.

6. COROLLARY. *Let (T, μ) be a σ-finite measure space, $G = L\{g\} \subset L_1(T, \mu)$ a one-dimensional linear subspace; the following two statements are equivalent*:

(a) *P_G is* l.s.c.

(b) *There does not exist an $f \in G^{\perp}\backslash\{0\}$ with*

(i) $S(f) \subset Z(g)$

(ii) supp(g) *is not the union of a finite number of atoms.*

Proof. We have to show only that condition (b) is equivalent to the formally weaker condition (b) of Theorem 3 (for one-dimensional G!); this is obvious, for, once we have found an $f \in G^{\perp}\backslash\{0\}$ with $S(f) \subset Z(g)$, we can change the values of f on $Z(g)$ in an appropriate way to get $\mu(S(f)) = 0$ without dropping the condition $\int_T f(t)g(t)\, d\mu(t) = 0$.

Corollary 4 contains [5, Satz 9]; Corollary 5 is identical with [5, Satz 12]; Corollary 6 contains [5, Satz 14].

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